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The asymptotic of static isolated systems and a generalized uniqueness for Schwarzschild

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Abstract

It has been proven that any static system that is spacetime-geodesically complete at infinity, and whose spacelike-topology outside a compact set is that of $\mathbb{R}^3$ minus a ball, is *asymptotically flat*. The matter is assumed to be compactly supported and no energy condition is required. A similar (though stronger) result also applies to black holes. This allows us to state a large generalization concerning the uniqueness of the Schwarzschild solution in not requiring asymptotic flatness. The Korotkin–Nicolai static black-hole shows that for the given generalization, no further flexibility in the hypothesis is possible.

Keywords: isolated systems, static solutions, asymptotic

1. Introduction

Asymptotic flatness is the basic notion used in general relativity (GR) to model systems that can be thought of as 'isolated' from the rest of the universe. It was used by Einstein himself, at least in heuristic form, and is now a standard piece of differential geometry and of gravitational and theoretical physics.

The notion of asymptotic flatness is also epistemologically linked to the Newtonian theory of gravitation[1]. In the 1916 manuscript *The Foundation of the Generalized Theory of Relativity*, Einstein addressed what he called *an epistemological defect* (but not mistake) of classical mechanics, whose origin he linked to Mach. He imagined two bodies, A and B, made of the same fluid material and sufficiently separated from each other that none of the properties of one could be attributed to the existence of the other. Observers at rest in one body see the other body rotating at a constant angular velocity, yet these same observers

[1] The following passage is based on a text I prepared for a highlight in CQG+.

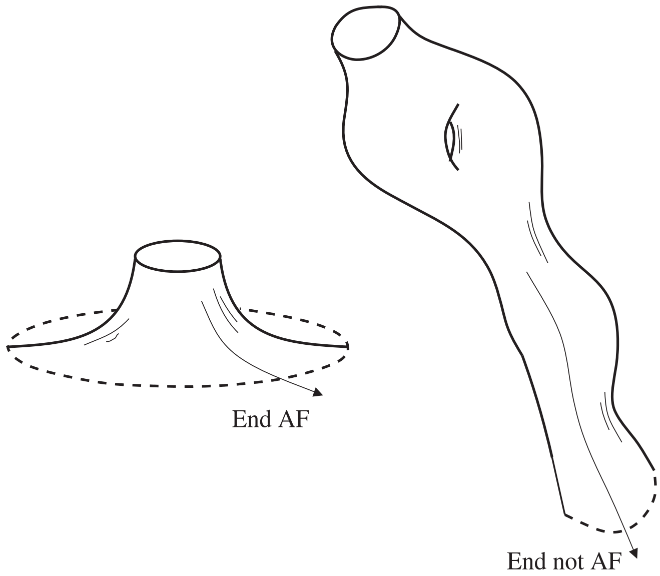

Figure 1. Representation of an AF end and a non-AF end.

measure a perfectly round surface in one case and an ellipsoid of rotation in the other case. He then asked: 'Why is this difference between the two bodies?'. Necessarily, he continues, the answer cannot be found inside the system A + B only; it must lie exterior to the system: the outer empty space. Einstein found that the source of the peculiar disparity was omitting that the empty space should also obey physical laws. These laws, which treat parts A and B of the system A + B + Exterior Empty Space on an equal footing, are the Einstein equations of GR. There is one point in Einstein's elegant conclusions that remains slightly inconclusive. It can be argued on the basis of GR that the absolute space of the 18th and 19th centuries was an inevitable concept, as 'corrections' to the Newtonian gravity are simply too small. Although this is unquestionable, it can also be demanded of GR to explain why this 'background solution', representing the Exterior Empty Space of the system described earlier, is so distinguished in a theory that treats the geometry and the asymptotic of space essentially as a variable.

Despite their importance, mathematical analysis of these questions was only recently initiated. The first general result came in 2000 with Anderson's *uniqueness* of the Minkowski spacetime [1] [2]. The result says that a geodesically complete static spacetime with no material sources (i.e. vacuum) is flat, and therefore covered by the Minkowski spacetime. Thus, the only geodesically complete and simply connected solution of the static Einstein equations empty of matter is Minkowski. This nicely illustrates the distinguished place that the Minkowski spacetime has among the physically relevant solutions. Around the same time, Anderson [2] started a systematic analysis of the global geometry of geodesically incomplete spacetimes including, for instance, spacetimes with boundary or singularities, and sufficient conditions for AF were given. The conclusions of [2] apply directly to spacetimes with compact sources as one can always excise a region containing matter and restrict the attention to the resulting space. More recently, necessary and sufficient conditions for asymptotic

[2] The results in [1] apply to strictly stationary solutions as well. In this article, we will refer only to static solutions.

flatness were investigated again in [8, 9] (see also [10]). We explain these results in some detail as they will be relevant for the rest of the paper. We begin with a formal definition of the vacuum static data set that will be useful later.

Definition 1.1. A static vacuum data set $(\Sigma; g, N)$ consists of a smooth three-manifold Σ, possibly with boundary, a smooth Riemannian metric g, and a smooth function N, such that,

(i) $(\Sigma; g)$ is metrically complete,
(ii) N is strictly positive in the interior Σ° of Σ, and
(iii) (g, N) satisfies the vacuum static Einstein equations,

$$N\mathrm{Ric} = \nabla\nabla N, \qquad \Delta N = 0. \tag{1.1}$$

Note that if N vanishes somewhere, then it does so only in points at the boundary of Σ (though N could also be strictly positive there). For instance, Σ can be part of a larger space after removing a region containing the sources (if any). The results in [8] and [9] concern the asymptotic of the ends of Σ and are independent of the geometry of the state in the 'bulk' of the manifold (including the boundary). They are stated as follows. Suppose that a closed region Σ' in the interior Σ° of Σ is diffeomorphic to $\mathbb{R}^3$ minus an open three-ball $\mathbb{B}^3$. If the space $(\Sigma', g' = N^2 g)$ is metrically complete, then the space $(\Sigma'; g, N)$ is asymptotically flat with Schwarzschildian fall off. The analysis in [8, 9] is made using the conformal metric $g' = N^2 g$ because of the remarkable properties that this metric has. The Ricci curvature is $\mathrm{Ric}_{g'} = 2\nabla \ln N \otimes \nabla \ln N$ and, in particular, is non-negative. Moreover, as also shown in [1], the space (Σ', g') has quadratic curvature decay (from $\partial\Sigma'$) provided that it is metrically complete. If (Σ', g) is metrically complete and $N \geqslant N_0 > 0$, then (Σ', g') is metrically complete, but in principle there is no reason to assume the completeness of the second space without any assumption on N. In this article, we prove that the completeness of (Σ', g') (in a situation as described above) follows from the suitable and physically natural geodesic completeness (until the boundary) of the associated spacetime. The result is summarized in theorem 1.3 and says that isolated systems, as defined below, are indeed always asymptotically flat.

As should be clear to the reader, the definition below is intended to capture the intuitive notion of a physical isolated system, but of course without making any reference to the asymptotic. The definition is a bit formal but it will give a good mathematical frame to be used later.

Definition 1.2. A globally hyperbolic static space-time $(\mathbf{M}, \mathbf{g})$ is called a static isolated system if there is an open set $\mathbf{K}$ of $\mathbf{M}$ containing (if any) the material sources, such that,

(i) the region $\mathbf{M}\backslash\mathbf{K}$ is diffeomorphic to

$$\mathbb{R} \times \left(\mathbb{R}^3 \backslash \mathbb{B}^3\right), \tag{1.2}$$

and on this region the space-time metric is of the form

$$\mathbf{g} = -N^2 \mathrm{d}t^2 + g, \tag{1.3}$$

where the lapse $N > 0$ and the spatial metric g are t-independent (∂_t is the static Killing), and

(ii) $(\mathbf{M}\backslash\mathbf{K}, \mathbf{g})$ is geodesically complete until its boundary, namely, geodesics (of any spacetime character) at either end of its boundary or defined for infinite parametric time.

To illustrate this definition, the simplest example of a static isolated system that one can imagine is a static spherically symmetric star. In this case, the space-time $\mathbf{M}$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}^3$ and the material source (the star) is contained inside the open spacetime region $\mathbf{K} = \{p \in \mathbf{M}\colon 0 \leqslant r(p) < 2r^*\}$, where r is the areal coordinate and r^* is the radius of the star. This region is represented schematically in the left of figure 1. Outside this region $\mathbf{K}$, the spacetime is Schwarzschild and clearly satisfies (i) and (ii).

We comment on several aspects of the definition. The topological condition (1.2) in (i) of the definition above is the most natural if one is describing an astrophysically realistic system like a neutron star. On the other hand, the existence of a static Killing field does not automatically imply the existence of *global* coordinates where the spacetime metric takes the form of (1.3) (though locally this is always the case where $|\partial_t| \neq 0$). The problem of the global existence of such coordinates is difficult and will not be considered here. In this sense, (1.3) has to be considered as an assumption and not as a consequence of staticity.

The geodesic completeness until the boundary in (ii) is a necessary condition to ensure that (roughly speaking) the physical boundary of $\mathbf{M}\backslash\mathbf{K}$ is just $\partial\mathbf{K}$. Geodesics are either 'infinite' or they reach $\partial\mathbf{K}$. The information that will be crucial for proving the metric completeness of $(\Sigma' = \mathbb{R}^3\backslash\mathbb{B}^3, g' = N^2 g)$ (from which AF will follow as explained earlier) is that concerning the completeness of the geodesics that move further and further away from the boundary. This should become clear during the proofs later. Once more, we stress that as $\mathbf{M}\backslash\mathbf{K}$ is free of matter, the data set (N, g) on $\Sigma' = \{t = 0\} \cap (\mathbf{M}\backslash\mathbf{K})$ satisfies the *vacuum* static equations

$$N\mathrm{Ric} = \nabla\nabla N, \qquad \Delta N = 0. \tag{1.4}$$

From now on we will call (ii) simply *geodesic completeness at infinity*: This terminology is justified by the following fact: geodesic completeness until the boundary holds if every spacetime geodesic, whose projection into $\mathbb{R}^3\backslash\mathbb{B}^3$ leaves any compact set, is complete.

In this setup we prove the following.

Theorem 1.3. *Static isolated systems are asymptotically flat with Schwarzschildian fall off.*

This theorem is an expression of the remarkable consistency of GR as a physical theory and shows the inevitability of asymptotic flatness in certain contexts.

The definition of *Schwarzschildian fall off* that we use in this theorem (and also above) is the simplest one and refers to the decay of g and N on a (suitable) coordinate patch. Concretely, $|\partial^i(g - g_S)| + |\partial^i(N - N_S)| = \mathcal{O}(r^{-2-i})$, where $g_S = (1 + m/2r)^4(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2)$ and $N_S = (1 - 2m/r)/(1 + m/2r)$ are, respectively, the usual metric and lapse of the Schwarzschild solution. The Schwarzschildian fall off does not play any special technical role in this article but it is important to state, as we did in theorem 1.3, the type of decay that static isolated systems have.

Along the same lines as in theorem 1.3, we can generalize the celebrated uniqueness of the Schwarzschild solution (Israel [6] [3], Robinson [11], Bunting–Masood-Ul Alam [3]) to a uniqueness statement among a (*a priori*) much larger class of static solutions than those AF. Accordingly, we consider static solutions given by vacuum static data $(\Sigma; g, N)$, i.e. with

[3] The Israel breakthough in 1967 was the first uniqueness theorem for Schwarschild and required that N could be chosen as a global radial coordinate.

$$N\mathrm{Ric} = \nabla\nabla N, \qquad \Delta N = 0, \tag{1.5}$$

and with a compact but not necessarily connected *horizon* $\partial\Sigma = \{N = 0\} \neq \emptyset$. As stated earlier, the solutions are said to be *geodesically compete at infinity* if spacetime geodesics, of any spacetime character, either end at the horizon (i.e. the boundary) or are defined for infinite parametric time.

The uniqueness theorem is the following.

Theorem 1.4. *Let* $(\Sigma; g, N)$ *be the data set of a static vacuum spacetime with a compact horizon and which is geodesically complete at infinity. Then, the spacetime is Schwarzschild if a connected component of the complement of an open set of* Σ *containing the boundary is diffeomorphic to* $\mathbb{R}^3\backslash\mathbb{B}^3$.

Essentially, the AF hypothesis that was required in earlier versions of the uniqueness theorem can be replaced by a necessary and sufficient topological condition.

Observe that in this statement nothing is said about the other connected components (if any) of the complement of the compact set. In principle, there could be many other unbounded connected components. That this cannot happen must be discerned after some analysis. This is similar in spirit to 'topological censorship'—the same type of theorems as in [4], although our technique is different as we cannot rely on any given structure at infinity.

To understand the importance and scope of this theorem, let us consider two purely relativistic examples. The first is of course the Schwarzschild black hole. It is a static vacuum solution with a topological-spherical hole, its curvature decays to zero at infinity, and the spacetime is geodesically complete at infinity. However (though not always properly emphasized), Schwarzschild is not the only static vacuum black hole solution in 3 + 1 dimensions enjoying these attributes. The other solution we are referring to is the Korotkin–Nicolai static black hole [7]. It represents a topologically spherical hole that is not inside an open (infinite) three-ball $\mathbb{B}^3$ as in Schwarzschild, but inside an open (infinite) solid-torus $\mathbb{B}^2 \times \mathbb{S}^1$. It is axially symmetric and has the asymptotic of a static Kasner [7] spacetime. Its space is not simply connected; for this reason, the horizon is prolate, as it feels the influence of itself along an axis of symmetry of finite length. The particular Kasner asymptotic is the simultaneous result of the presence of the hole on one side and of the non-trivial global topology on the other. Finite covers of the solution yield static spacetimes with a finite number of black holes in equilibrium. From the point of view of the general theory of relativity, the Korotkin–Nicolai and Schwarzschild solutions are perfectly acceptable, although one is AF and the other is not. This shows that the topological assumption in theorem 1.4 cannot be eliminated altogether.

In parallel to the discussion given at the beginning of the introduction, it is worth noting that theorem 1.3 can be interpreted as a result of 'asymptotic uniqueness' (here asymptotic flatness), and that, in this sense, it is a close relative of the uniqueness of the flat Minkowski spacetime among complete (simply connected) vacuum static spacetimes as proved by Anderson in [1]. Anderson's result is a direct consequence of a curvature decay that we will explain in section 2.1. We stress, however, that such decay is not nearly sufficient to deduce asymptotic flatness. The Korotkin–Nicolai solution satisfies this curvature decay and is not AF.

The rest of the article is roughly organized as follows. Sections 2.1, 2.2 and 2.3 deal with some important facts about the global structure of the vacuum static solutions. Section 3 contains the proofs of theorems 1.3 and 1.4. Proposition 3.1 shows the existence of a natural partition of static ends of the form $\mathbb{R}^3\backslash\mathbb{B}^3$. Proposition 3.2 then proves that the lapse N can have only three types of behaviours at infinity and proposition 3.3 proves the completeness of $N^2 g$ on the end. The proof of theorems 1.3 and 1.4 are given afterwards.

2. Background material

A smooth Riemannian metric g on a smooth connected manifold Σ (with or without boundary, compact or not) induces the metric

$$\mathrm{dist}(p, q) = \inf\left\{ \mathrm{length}\left(\gamma_{pq}\right): \gamma_{pq} \text{ smooth curve joining } p \text{ to } q \right\}. \tag{2.1}$$

The space $(\Sigma; g)$ is said to be *metrically complete* if $(\Sigma; \mathrm{dist})$ is complete. If Σ has a compact boundary, then metric completeness is equivalent to the *geodesic completeness until the boundary* of $(\Sigma; g)$ (by Hopf–Rinow). On the other hand, geodesics in $(\Sigma; g)$ lift to geodesics perpendicular to the static Killing field in the associated spacetime, i.e. in

$$\mathbf{M} = \mathbb{R} \times \Sigma, \qquad \mathbf{g} = -N^2 \mathrm{d}t^2 + g. \tag{2.2}$$

Hence, if $\partial\Sigma$ is compact, then geodesic completeness until the boundary of $(\mathbf{M}; \mathbf{g})$ implies the metric completeness of $(\Sigma; g)$. This is used in proposition 3.3.

Geodesic completeness until the boundary of $(\mathbf{M}; \mathbf{g})$ is a basic assumption in the two main theorems in this article. However, regarding possible mathematical applications, it is important to assume only the metric completeness of the data whenever possible. We will make some remarks in this respect.

If $\partial\Sigma \neq \emptyset$, we define the *metric annulus* $\mathcal{A}(a, b)$ of radii $0 < a < b$ by

$$\mathcal{A}(a, b) = \{p \in \Sigma: a < \mathrm{dist}(p, \partial\Sigma) < b\}, \tag{2.3}$$

where $\mathrm{dist}(p, \partial\Sigma) = \inf\{\mathrm{dist}(p, q): q \in \partial\Sigma\}$.

2.1. Anderson's curvature decay

Anderson's curvature decay [1] is an important property of static solutions. It says that there is a universal constant $\eta > 0$ such that for any static data $(\Sigma; g, N)$, we have

$$|\mathrm{Ric}|(p) \leqslant \frac{\eta}{\mathrm{dist}_g^2(p, \partial\Sigma)}, \quad \text{and} \quad \left|\frac{\nabla N}{N}\right|^2 (p) \leqslant \frac{\eta}{\mathrm{dist}_g^2(p, \partial\Sigma)}. \tag{2.4}$$

The optimal constant η can be seen to be greater than or equal to one, but it is not know if it is one. Upper bounds can be given, but they are far from one.

As an application of the curvature decay, let us prove here the proposition that will be used in the proof of theorem 1.4 to rule multiple ends when it is known that there is one that is AF. In the statement, we use Σ_δ to denote the manifold resulting from removing from Σ the tubular neighbourhood of $\partial\Sigma$ and radius δ, i.e. $\Sigma_\delta = \Sigma \setminus \{p: \mathrm{dist}_g(p, \partial\Sigma) < \delta\}$. We assume that $\delta < \delta_0$ with δ_0 small enough that $\partial\Sigma_\delta$ is always smooth.

Proposition 2.1. *Let $(\Sigma; g, N)$ be a static vacuum initial data set with a compact horizon $(\partial\Sigma = \{N = 0\} \neq \emptyset)$ and $(\Sigma; g)$ be metrically compete. Then there is $0 < \epsilon_0 < 1$ such that for every $\epsilon < \epsilon_0$ there is $\delta < \delta_0$, such that $(\Sigma_\delta; N^{-2\epsilon} g)$ is metrically complete and $\partial\Sigma_\delta$ is strictly convex (with respect to the outward normal).*

Proof. Given $0 < \epsilon < 1$, the convexity of $\partial\Sigma_\delta$ for small enough $\delta \leqslant \delta_0$ is direct (and we leave it to the reader) as the factor $N^{-2\epsilon}$ 'blows up' the boundary $\partial\Sigma$ uniformly (observe, however, that as $\epsilon < 1$, $\partial\Sigma$ 'remains' at a finite distance from the bulk of Σ).

So let us prove that if we chose small enough ϵ, the space $(\Sigma_\delta, N^{-2\epsilon}g)$ is metrically complete. As we assume $\delta < \delta_0$, it is sufficient to prove that if ϵ is small enough, then $(\Sigma_{\delta_0}, N^{-2\epsilon}g)$ is metrically complete[4]. We will do that below, and the argument is thus independent of δ.

It is enough to prove that if ϵ is small enough, then the following holds: for any sequence of points p_i whose g distance to $\partial\Sigma_{\delta_0}$ diverges, the $(N^{-2\epsilon}g)$ distance to $\partial\Sigma_{\delta_0}$ also diverges. Equivalently, it is sufficient to prove that for any sequence of curves γ_i starting at $\partial\Sigma_{\delta_0}$ and ending at p_i, we have

$$\int_0^{s_i} \frac{1}{N^\epsilon\left(\gamma_i(s)\right)} \mathrm{d}s \longrightarrow \infty, \tag{2.5}$$

where s is the g arc length of γ_i starting from $\partial\Sigma_{\delta_0}$. Now, as we will show below, the curvature decay (2.4) immediately implies the estimate

$$N(p) \leqslant c\left(1 + \mathrm{dist}_g\left(p, \partial\Sigma_{\delta_0}\right)\right)^{\sqrt{\eta}} \tag{2.6}$$

for any $p \in \Sigma$, where $\eta > 0$ is universal but c depends on (Σ, g) and δ_0. As $\mathrm{dist}_g(\gamma_i(s), \partial\Sigma_{\delta_0}) \leqslant s$, then we have

$$N\left(\gamma_i(s)\right) \leqslant c(1 + s)^{\sqrt{\eta}}. \tag{2.7}$$

Thus, if $\epsilon < 1/\sqrt{\eta}$, then

$$\int_0^{s_i} \frac{1}{N^\epsilon\left(\gamma_i(s)\right)} \mathrm{d}s \geqslant \int_0^{s_i} \frac{1}{c^\epsilon(1 + s)^{\epsilon\sqrt{\eta}}} \mathrm{d}s = \frac{1}{c^\epsilon\left(1 - \epsilon\sqrt{\eta}\right)}\left(\left(1 + s_i\right)^{1-\epsilon\sqrt{\eta}} - 1\right) \tag{2.8}$$

$$\geqslant \frac{1}{c^\epsilon\left(1 - \epsilon\sqrt{\eta}\right)}\left(\left(1 + \mathrm{dist}_g\left(p_i, \partial\Sigma_{\delta_0}\right)\right)^{1-\epsilon\sqrt{\eta}} - 1\right) \longrightarrow \infty \tag{2.9}$$

as wished.

Now we briefly comment on the derivation of (2.6). Let $\gamma(s)$ be a geodesic joining a point p to $\partial\Sigma_{\delta_0/2}$ and realizing the distance between p and $\partial\Sigma_{\delta_0/2}$. Let p_0 be the point of intersection between γ and Σ_{δ_0}. Then

$$\left|\ln \frac{N(p)}{N(p_0)}\right| = \left|\int_{\mathrm{dist}_g\left(p_0, \partial\Sigma_{\delta_0/2}\right)\left(=\delta_0/2\right)}^{\mathrm{dist}_g\left(p, \partial\Sigma_{\delta_0/2}\right)} \frac{\nabla_{\gamma'(s)}N}{N} \mathrm{d}s\right| \leqslant \int_{\delta_0/2}^{\mathrm{dist}_g\left(p, \partial\Sigma_{\delta_0/2}\right)} \frac{\sqrt{\eta}}{s} \mathrm{d}s \leqslant \tag{2.10}$$

$$\leqslant \ln\left(\frac{\mathrm{dist}\left(p, \partial\Sigma_{\delta_0/2}\right)}{\delta_0/2}\right)^{\sqrt{\eta}}, \tag{2.11}$$

[4] As $\Sigma_\delta = \Sigma_{\delta_0} \cup \overline{(\Sigma_\delta \backslash \Sigma_{\delta_0})}$ and $\overline{(\Sigma_\delta \backslash \Sigma_{\delta_0})}$ is a smooth compact manifold, the manifold $(\Sigma_\delta, N^{-2\epsilon}g)$ is metrically complete if $(\Sigma_{\delta_0}, N^{-2\epsilon}g)$ is metrically complete.

and hence

$$N(p) \leqslant N(p_0)\left(\frac{\text{dist}\left(p,\, \partial\Sigma_{\delta_0/2}\right)}{\delta_0/2}\right)^{\sqrt{\eta}}. \tag{2.12}$$

One then uses the general estimates $\text{dist}_g(p, \partial\Sigma_{\delta_0}/2) \leqslant \text{dist}_g(p, \Sigma_{\delta_0}) + \delta_0/2$ and $N(p_0) \leqslant \max\{N(q): q \in \partial\Sigma_\delta\}$ to show easily (2.6) for a suitable c big enough. □

There are two properties about the spaces $(\Sigma_\delta, N^{-2\epsilon})$ that will be central in the proof of theorem 3.1 and that are worth highlighting here. First, the manifold $(\Sigma_\delta, N^{-2\epsilon}g)$ is geodesically convex, that is, any two points in Σ_δ° can be joined by a length-minimizing geodesic contained inside Σ_δ°. This is indeed a direct consequence of the strict convexity and compactness of the boundary $\partial\Sigma_\delta$[5]. In particular, if Σ has two ends, then so does Σ_δ, and one can guarantee the existence of a line diverging along the two ends. Second, but not less important, the Ricci curvature of the metric $\tilde{g} = N^{-2\epsilon}g$ has the expression[6]

$$\tilde{\text{Ric}} = -\tilde{\nabla}\,\tilde{\nabla}f + \frac{1}{c}\tilde{\nabla}f\,\tilde{\nabla}f, \tag{2.13}$$

where f and c depend on ϵ and are given by

$$f = -(1+\epsilon)\ln N, \quad \text{and} \quad \frac{1}{c} = \frac{\left(1 - 2\epsilon - \epsilon^2\right)}{(1+\epsilon)^2}. \tag{2.14}$$

In particular, if $0 < \epsilon < \sqrt{2} - 1$, then $c > 0$ and the c-Bakry–Emery Ricci tensor $\tilde{\text{Ric}}_f^c$, which is defined by

$$\tilde{\text{Ric}}_f^c = \tilde{\text{Ric}} + \tilde{\nabla}\,\tilde{\nabla}f - \frac{1}{c}\tilde{\nabla}f\,\tilde{\nabla}f, \tag{2.15}$$

is zero. In the proof of theorem 3.1, we will use these two observations together to make some simple mean-curvature comparisons (à la Bakry–Emery).

2.2. The ball covering property

As observed in [2], Liu's *ball covering property* holds for (metrically complete) static solutions $(\Sigma; g)$ with compact boundaries. Namely, for any $0 < a < b$, there is r_0 and n_0 such that for any $r \geqslant r_0$ there is a set of balls $\{B(p_i, ar/2), p_i \in \overline{\mathcal{A}}(ar, br), i = 1, \ldots, n_r \leqslant n_0\}$ covering $\overline{\mathcal{A}}(ar, br)$. Here and below $\overline{\mathcal{A}}$ is the closure of $\mathcal{A}$.

As a direct corollary, we see that for any $0 < a < b$ and $r \geqslant r_0$, as in the ball covering property, any two points in the same connected component of $\mathcal{A}(ar, br)$ can be joined by a curve of length less than or equal to $n_0 ar$ entirely contained in $\mathcal{A}(ar/3, 3br)$.

[5] Observe that a length-minimizing sequence of curves (with fixed end-points) must remain at a definite distance away from the boundary, as otherwise their lengths could be reduced in a definite amount (due to the strict convexity). With this property granted, the limit of the sequence (or of a subsequence if necessary) must be a geodesic in Σ_δ° by standard arguments.

[6] For this, if $\tilde{g} = e^{2\phi}g$, then $\tilde{\text{Ric}} = \text{Ric} - (\nabla\nabla\phi - \nabla\phi\nabla\phi) - (\Delta\phi + |\nabla\phi|^2)g$ and $\tilde{\nabla}_i V_j = \nabla_i V_j - (V_j\nabla_i\phi + V_i\nabla_j\phi - (V^k\nabla_k\phi)g_{ij})$.

Let $\mathcal{A}_c(ar, br)$ be a connected component of $\mathcal{A}(ar, br)$. By the curvature decay (2.4), we have $|\nabla N/N| \leqslant 3\eta/ar$ all over $\mathcal{A}_c(ar/3, 3br)$. By integrating this inequality along curves as in the previous paragraph, we obtain[7]

$$\frac{\max\left\{N(p)\colon p \in \overline{\mathcal{A}}_c(ar, br)\right\}}{\min\left\{N(p)\colon p \in \overline{\mathcal{A}}_c(ar, br)\right\}} \leqslant C(a, b). \tag{2.16}$$

This is a type of Harnack inequality for N and is fundamental.

Remark 2.2. It is not known at the moment if a similar ball covering property holds for strictly stationary solutions. This is a main obstacle to extending theorem 1.3 to stationary isolated systems.

2.3. Spacetime geodesics in static spacetimes

Let $(\Sigma; g, N)$ be a static vacuum data and let $(\mathbf{M}, \mathbf{g})$ be its associated spacetime. We recall here a useful way to describe spacetime geodesics $\Gamma(\tau)$ in terms of certain metrics conformal to g in Σ. This goes back at least to the work of Weyl [13] from 1917.

Let $\gamma = \Pi(\Gamma)$ be the projection of Γ into Σ. Then it is easy to see that γ satisfies the equation

$$\nabla_{\gamma'}\gamma' = -a^2\frac{\nabla N}{N^3}, \tag{2.17}$$

where $\gamma' = \mathrm{d}\gamma/\mathrm{d}\tau$ and a is the constant $a = \mathbf{g}(\Gamma', \partial_t)$. Moreover, we have

$$|\gamma'|^2 = \varepsilon + \frac{a^2}{N^2}, \tag{2.18}$$

where the norm on the left-hand side is with respect to g and $\varepsilon = -1, 0, 1$ according to the character type of the geodesic.

Then define $\mathrm{e}^{2\phi}$ by

$$\mathrm{e}^{2\phi} = \left(\varepsilon + \frac{a^2}{N^2}\right), \tag{2.19}$$

wherever the right-hand side is positive (this includes the projection of the geodesic). Finally, we consider the conformal metrics

$$\hat{g} = \mathrm{e}^{2\phi}g, \qquad \check{g} = \mathrm{e}^{-2\phi}g, \tag{2.20}$$

and use $\mathrm{d}s$, $\mathrm{d}\hat{s} = \mathrm{e}^{\phi}\mathrm{d}s$, and $\mathrm{d}\check{s} = \mathrm{e}^{-\phi}\mathrm{d}s$ to denote the elements of length of γ with respect to g, $\hat{g}$ and $\check{g}$, respectively.

In this setup, we have the following characterization: if $\Gamma(\tau)$ is a spacetime geodesic, then $\gamma(\hat{s})$ is a geodesic of $\hat{g}$ and $\mathrm{d}\tau = \mathrm{d}\check{s}$. Conversely, if $\gamma(\hat{s})$ is a geodesic of $\hat{g}$, then the curve

$$\Gamma\left(\check{s}\right) = \left(\int^{\check{s}} \frac{-a}{N^2\left(\gamma\left(\check{s}'\right)\right)}\mathrm{d}\check{s}', \gamma\left(\check{s}\right)\right) \subset \mathbb{R} \times \Sigma = \mathbf{M} \tag{2.21}$$

is a spacetime geodesic with $\mathbf{g}(\Gamma', \Gamma') = \varepsilon$, and hence with $\tau = \check{s}$.

[7] If $\gamma(s)$ is a curve of length less than or equal to $n_0 ar$ joining the points p_1 and p_2, then $|\ln N(p_1)/N(p_2)| = |\int(\nabla_{\gamma'}N)/N\ \mathrm{d}s| \leqslant (3\eta/ar)n_0 ar = 3n_0\eta$. From this we deduce $\mathrm{e}^{-3n_0\eta} \leqslant N(p_2)/N(p_1) \leqslant \mathrm{e}^{3n_0\eta}$ and (2.16) follows (note $n_0 = n_0(a, b)$).

Two points are particularly important about this characterization of spacetime geodesics: (i) spacetime geodesics can be constructed out of the projected curves which in turn can be easily found through length-minimization, and (ii) as the affine parameter of spacetime geodesics is the $\breve{g}$-arc length of the projected curve, there is a way to link spacetime geodesic completeness at infinity to the metric completeness of $\breve{g} = N^2 g$. We will exploit these two observations during the proof of proposition 3.3. We will only use the characterization of null geodesics, i.e. $\epsilon = 0$, although other types of geodesics can be useful in similar contexts.

3. The proofs

Every smooth, connected, compact, boundaryless and orientable surface F embedded in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into two connected components. Below we will work with such surfaces F embedded in $\mathbb{R}^3 \setminus \mathbb{B}^3$ and will denote by $M(F)$ the closure of the bounded connected component of $(\mathbb{R}^3 \setminus \mathbb{B}^3) \setminus F$. Two facts are simple to check. First, for any disjoint F_1 and F_2 such that $\partial\mathbb{B}^3 \subset M(F_i)$ for $i = 1, 2$, either $F_1 \subset M^\circ(F_2)$ or $F_2 \subset M^\circ(F_1)$ (here $^\circ$ = interior). Second, if a set $\{F_i,\ i = 1, \ldots, n \geqslant 1\}$ of such surfaces is such that $\partial\mathbb{B}^3$ belongs to a bounded component of $\Sigma \setminus \bigcup_{i=1}^{i=n} F_i$ then there is at least one F_i such that $\partial\mathbb{B}^3 \subset M(F_i)$. We will use these facts in the proof of the following proposition.

Proposition 3.1. *Let $(\Sigma; g, N)$ be a metrically complete vacuum static data set with $\Sigma \approx \mathbb{R}^3 \setminus \mathbb{B}^3$. Then, there is a set of (smooth, connected, compact, boundaryless and orientable) surfaces $\{S_j;\ j = 0, 1, 2, 3, \ldots\}$, such that the following holds for every j:*

1. *S_{j} is embedded in $\mathcal{A}(2^{1+2j}, 2^{2+2j})$,*
2. *$\partial\Sigma \subset M(S_j)$, and*
3. *$M(S_j) \subset M(S_{j+1})$.*

The surfaces S_i will be used only as references inside the manifold Σ; their geometries play no role. Observe that $\Sigma \setminus M(S_k) = \bigcup_{j=k}^{j=\infty} M(S_{j+1}) \setminus M(S_j)$ with the union disjoint and that $S_{j+1} \cup S_j = \partial(M(S_{j+1}) \setminus M^\circ(S_j))$. This last observation will be used when we apply the maximum principle to N on $M(S_{j+1}) \setminus M^\circ(S_j)$.

Proof. In the argument that follows, we treat Σ and $\mathbb{R}^3 \setminus \mathbb{B}^3$ indistinctly. The construction of the surfaces S_j, $j = 0, 1, 2, \ldots$ is as follows. Let $f\colon \Sigma \to [0, \infty)$ be a (any) smooth function such that $f \equiv 1$ on $\{p\colon \mathrm{dist}(p, \partial\Sigma) \leqslant 2^{1+2j}\}$ and $f \equiv 0$ on $\{p\colon \mathrm{dist}(p, \partial\Sigma) \geqslant 2^{2+2j}\}$. Let x be any regular value of f in (0, 1). Then we can write $f^{-1}(x) = F_1 \cup \cdots \cup F_n$, where each F_i is a (connected, compact, boundaryless and orientable) surface embedded in $\mathcal{A}(2^{1+2j}, 2^{2+2j})$. Now, as Σ is the disjoint union of the sets $f^{-1}((x, \infty))$, $f^{-1}(x) = \bigcup_{i=1}^{i=\infty} F_i$ and $f^{-1}((-\infty, x))$, and as $\{p\colon \mathrm{dist}(p, \partial\Sigma) \geqslant 2^{2+2j}\} \subset f^{-1}((-\infty, x))$, we conclude that $\partial\Sigma$, which lies inside $f^{-1}((x, \infty))$, must belong to a bounded component of $\Sigma \setminus \bigcup_{i=1}^{i=n} F_i$. Hence $\partial\Sigma \subset M(F_{i*})$ for some F_{i*}, (see the beginning of this section). We set $S_j = F_{i*}$.

We now verify that the surfaces S_j satisfy properties 1–3. By construction the S_js already satisfy 1 and 2. Now, either $M(S_j) \subset M^\circ(S_{j+1})$ or $M(S_{j+1}) \subset M^\circ(S_j)$. If $M(S_{j+1}) \subset M^\circ(S_j)$, then $S_{j+1} \subset \{p\colon \mathrm{dist}(p, \partial\Sigma) < 2^{2+2j}\}$, which is impossible because $S_{j+1} \subset \mathcal{A}(2^{3+2j}, 2^{4+2j})$. Thus, $M(S_j) \subset M^\circ(S_{j+1})$, showing property 3. □

We claim that for any $j \geqslant 0$, the surfaces S_{j+1} and S_j lie in the same connected component of the annuli $\mathcal{A}(2^{1+2j}, 2^{4+2j})$. To see this, consider a ray $\gamma(s)$, $s \geqslant 0$, starting at $\partial\Sigma$ at $s = 0$, (i.e. $\mathrm{dist}(\gamma(s), \partial\Sigma) = s$ for all $s \geqslant 0$; s is arc-length). Let s_j be the last time that $\gamma(s) \in S_j$ and let s_{j+1} be the first time that $\gamma(s) \in S_{j+1}$. Then, $s_j \geqslant 2^{1+2j}$ because $S_j \subset \mathcal{A}(2^{1+2j}, 2^{2+2j})$ and $s_{j+1} \leqslant 2^{4+2j}$ because $S_{j+1} \subset \mathcal{A}(2^{3+2j}, 2^{4+2j})$. Hence the arc $\{\gamma(s): s \in [s_j, s_{j+1}]\}$ must lie inside $\mathcal{A}(2^{1+2j}, 2^{4+2j})$ because $\mathrm{dist}(\gamma(s), \partial\Sigma) = s$ for all s. We then conclude that S_j and S_{j+1} must lie in the same connected component of $\mathcal{A}(2^{1+2j}, 2^{4+2j})$.

This claim and proposition 3.1 will be used in the proof of the following proposition.

Proposition 3.2. *Let $(\Sigma; g, N)$ be a metrically complete vacuum static data set with $\Sigma \approx \mathbb{R}^3 \backslash \mathbb{B}^3$ and $N > 0$. Then, one of the following holds:*

1. *N converges uniformly to zero over the end of Σ,*
2. *N converges uniformly to infinity over the end of Σ,*
3. *$C_1 < N < C_2$ for constants $0 < C_1 < C_2 < \infty$.*

Proof. To shorten notation, we will write $\max\{N; \Omega\}$: $\max\{N(p): p \in \Omega\}$, where Ω are compact sets (same notation for $\min\{N; \Omega\}$).

Suppose that there is a divergent sequence p_i for which $N(p_i) \to 0$ as $i \to \infty$. We claim that, in this case, N tends uniformly to zero over the end.

For every i let j_i be such that $p_i \in M(S_{j_i}) \backslash M^\circ(S_{j_i - 1})$. Suppose first that

$$\max\left\{N; S_{j_i}\right\} \to 0. \tag{3.1}$$

Then, for any $i' > i$, the maximum principle gives

$$\max\left\{N; M\left(S_{j_{i'}}\right)\backslash M^\circ\left(S_{j_i}\right)\right\} \leqslant \max\left\{\max\left\{N; S_{j_{i'}}\right\}, \max\left\{N; S_{j_i}\right\}\right\}. \tag{3.2}$$

Letting $i' \to \infty$ and using (3.1), we obtain

$$\sup\left\{N(p): p \in \Sigma\backslash M^\circ\left(S_{j_i}\right)\right\} \leqslant \max\left\{N; S_{j_i}\right\}, \tag{3.3}$$

where the right-hand side tends to zero as i tends to infinity. This proves that N tends uniformly to zero as claimed.

To prove (3.1), we recall, as was pointed out earlier, that S_{j_i} and $S_{j_i - 1}$ lie in the same connected component of $\mathcal{A}(2^{2j-1}, 2^{2j+2})$. Observe too that the annuli $\mathcal{A}(2^{2j-1}, 2^{2j+2})$ can be written as $\mathcal{A}(ar_i br_i)$ with $a = 1/2$, $b = 4$ and $r_i = 2^{2j_i}$. Therefore, we can use the discussion of section 2.2 to deduce that

$$\max\left\{N; S_{j_i}\right\} \leqslant c \min\left\{N; S_{j_i} \cup S_{j_i - 1}\right\}, \tag{3.4}$$

where the constant c is independent of i. On the other hand, by the maximum principle, we have

$$\min\left\{N; S_{j_i} \cup S_{j_i - 1}\right\} \leqslant \min\left\{N; M\left(S_{j_i}\right)\backslash M^\circ\left(S_{j_i - 1}\right)\right\} \leqslant N(p_i). \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\max\left\{N;\, S_{j_i}\right\} \leqslant N(p_i), \tag{3.6}$$

where the right-hand side tends to zero. This implies (3.1) as desired.

In the same manner, one proves that if there is a divergent sequence p_i such that $N(p_i) \to \infty$ as $i \to \infty$, then N tends uniformly to infinity over the end.

If none of the situations considered above occur, then $0 < C_1 < N < C_2$ for the constants C_1, C_2. □

To show the asymptotic flatness for isolated systems using [8, 9], we need only to prove the completeness of N^2g using the assumption that the static spacetime is geodesically complete at infinity. This is done in the next proposition.

Proposition 3.3. *Let* $(\Sigma;\, g,\, N)$ *be a static vacuum data set, with* $\Sigma \approx \mathbb{R}^3\backslash\mathbb{B}^3$ *and* $N > 0$ *on* Σ. *Assume that the associated spacetime*

$$\mathbf{M} = \mathbb{R} \times \Sigma, \qquad \mathbf{g} = -N^2\mathrm{d}t^2 + g \tag{3.7}$$

is geodesically complete at infinity. Then the space $(\Sigma;\, N^2g)$ *is metrically complete.*

Proof. The proof is made by contradiction. So let us assume that $(\Sigma;\, N^2g)$ is not metrically complete. We will explain later how this contradicts the *geodesic completeness at infinity*. During the proof, we use the same notation as in proposition 3.2. We will also assume, as was explained in section 2, that under the hypothesis of the proposition, the space $(\Sigma;\, g)$ is metrically complete.

We begin by proving that

$$\sum_{j=1}^{j=\infty} \max\left\{N;\, S_j\right\}2^{2j} < \infty. \tag{3.8}$$

Let β: $[s_j,\, s_{j+1}] \to M(S_{j+1})\backslash M^\circ(S_j)$ be any curve with $\beta(s_j) \in S_j$ and $\beta(s_{j+1}) \in S_{j+1}$. We claim that then

$$\int_{s_j}^{s_{j+1}} N(\beta(s))\mathrm{d}s \geqslant c_1 \max\left\{N;\, S_j\right\}2^{2j}, \tag{3.9}$$

where the constant c_1 is independent of j. To see this, we write

$$\int_{s_j}^{s_{j+1}} N(\beta(s))\mathrm{d}s \geqslant \min\left\{N;\, M\left(S_{j+1}\right)\backslash M^\circ\left(S_j\right)\right\}\mathrm{length}(\beta) \tag{3.10}$$

and note that

1. $\mathrm{length}(\beta) \geqslant (2^{3+2j} - 2^{1+2j}) = 6\, 2^{2j}$, because it is $S_j \subset \mathcal{A}(2^{1+2j},\, 2^{2+2j})$, and $S_{j+1} \subset \mathcal{A}(2^{3+2j},\, 2^{4+2j})$,
2. $\min\{N;\, M(S_{j+1})\backslash M^\circ(S_j)\} \geqslant \max\{N;\, S_j\}$, because

$$\min\left\{N;\, M\left(S_{j+1}\right)\backslash M^\circ\left(S_j\right)\right\} \geqslant \min\left\{N;\, S_{j+1} \cup S_j\right\} \tag{3.11}$$

by the maximum principle, and because

$$\min\left\{N;\, S_{j+1} \bigcup S_j\right\} \geqslant c_2 \max\left\{N;\, S_j\right\}, \tag{3.12}$$

where c_2 is independent of j, as was explained in section 2.2,[8].

The formula (3.9) is then obtained making $c_1 = 6c_2$.

Now, if $(\Sigma;\, N^2 g)$ is not metrically complete, then one can find a sequence of points p_i, with $\mathrm{dist}_g(p_i,\, \partial\Sigma) \to \infty$ but with $\mathrm{dist}_{N^2 g}(p_i,\, \partial\Sigma)$ uniformly bounded. From the definition of dist, this implies that there is a sequence of curves $\alpha_i(s)$; $s \in [0,\, s_i]$ starting at $\partial\Sigma$ and ending at p_i, for which

$$\int_{s=0}^{s=s_i} N(\alpha(s))\mathrm{d}s \leqslant K < \infty, \tag{3.13}$$

where K is independent of j. For every i let j_i be the greatest j such that $p_i \notin M(S_j)$. Then, for every $j \leqslant j_i - 1$ one can find an interval $[s_{j,i},\, s_{j+1,i}]$ such that the curve β_j defined by $\beta_j(s) = \alpha_i(s)$, $s \in [s_{j,i},\, s_{j+1,i}]$, has range in $M(S_{j+1})\backslash M^\circ(S_j)$ and moreover with $\beta_j(s_{j,i}) \in S_j$ and $\beta_j(s_{j+1,i}) \in S_{j+1}$. Using (3.9) we write

$$K \geqslant \int_{s=0}^{s=s_i} N(\alpha)\mathrm{d}s \geqslant \sum_{j=1}^{j=j_i-1} \int_{s_{j,i}}^{s_{j+1,i}} N\left(\beta_j\right)\mathrm{d}s \geqslant \sum_{j=1}^{j=j_i-1} c_1 \max\left\{N;\, S_j\right\} 2^{2j}. \tag{3.14}$$

Taking the limit $i \to \infty$ gives (3.8) as wished.

We proceed now with the proof. By proposition 3.2 we know that N must go uniformly to zero at infinity otherwise N would be bounded from below away from zero and the metric $N^2 g$ would be automatically complete. If $N \to 0$ uniformly at infinity, then $(\Sigma;\, N^{-2} g)$ is metrically complete.

As was explained in section 2.3, null-spacetime geodesics project into $(N^{-2} g)$ geodesics and the affine parameter is the $(N^2 g)$ arc length. We will see below that if $(\Sigma;\, N^2 g)$ is not metrically complete, then there is an infinite $(N^{-2} g)$ geodesic whose $(N^2 g)$ length is finite. This would be against the hypothesis that the spacetime is geodesically complete at infinity and the proof will be finished.

Let $\Gamma(s)$, $s \geqslant 0$ be a ray for the metric to $N^{-2} g$ starting at $\partial\Sigma$. For each $j \geqslant 1$, let s_j be the last time that $\Gamma(s) \in S_j$. Let Γ_j be the restriction of Γ to $[s_j,\, s_{j+1}]$. Then $\Gamma_j \subset (\Sigma\backslash M^\circ(S_j))$ and Γ is the concatenation of the curves Γ_j, $j \geqslant 1$. Now,

$$\int_{s=s_1}^{s=\infty} N(\Gamma(s))\mathrm{d}s = \sum_{j=1}^{j=\infty} \int_{s_j}^{s_{j+1}} N\left(\Gamma_j(s)\right)\mathrm{d}s \leqslant \sum_{j=1}^{j=\infty} \max\left\{N;\, S_j\right\} \mathrm{length}\left(\Gamma_j\right), \tag{3.15}$$

where to obtain the inequality we use

$$\sup\left\{N\left(\Gamma_j(s)\right):\, s \in [s_j,\, s_{j+1}]\right\} \leqslant \sup\left\{N(p):\, p \in \Sigma\backslash M^\circ\left(S_j\right)\right\} \leqslant \max\left\{N;\, S_j\right\}, \tag{3.16}$$

which is obtained from the inclusion $\Gamma_j \subset (\Sigma\backslash M^\circ(S_j))$ (for the first inequality) and the maximum principle (for the second). Thus, if we prove that for a constant c_3 independent of j we have

$$\mathrm{length}\left(\Gamma_j\right) \leqslant c_3 2^{2j}, \tag{3.17}$$

[8] As in the proof of proposition 3.2, recall that S_j and S_{j+1} lie in the same connected component of $\mathcal{A}(2^{1+2j},\, 2^{3+2j})$ (see remark after the proof of proposition 3.1), and observe too that we can write $\mathcal{A}(2^{1+2j},\, 2^{3+2j}) = \mathcal{A}(ar_j,\, br_j)$ with $a = 2$, $b = 16$ and $r_j = 2^{2j}$).

then we can use (3.8) in conjunction to (3.15) to conclude that

$$\int N(\Gamma(s))\mathrm{d}s < \infty \tag{3.18}$$

which would imply that there is an incomplete null geodesic in the spacetime.

Let us prove then the inequality (3.17). We will play with the fact that Γ is a ray for $N^{-2}g$.

First, note

$$\int_{s_j}^{s_{j+1}} \frac{1}{N\left(\Gamma_j(s)\right)}\mathrm{d}s \geqslant \frac{\mathrm{length}\left(\Gamma_j\right)}{\max\left\{N;\, \Gamma_j\right\}} \geqslant \frac{\mathrm{length}\left(\Gamma_j\right)}{\max\left\{N;\, S_j\right\}}, \tag{3.19}$$

where the second inequality is obtained from the inclusion $\Gamma_j \subset \Sigma\backslash M^\circ(S_j)$ and because $\max\{N;\, \Sigma\backslash M(S_j)\} \leqslant \max\{N;\, S_j\}$ by the maximum principle.

Then recall from the discussion after proposition 3.1 that S_j and S_{j+1} lie in the same connected component $\mathcal{A}_c(2^{1+2j}, 2^{4+2j})$ of $\mathcal{A}(2^{1+2j}, 2^{4+2j})$. Hence, $\Gamma(s_j)(\in S_j)$ and $\Gamma(s_{j+1})(\in S_{j+1})$ also lie in $\mathcal{A}_c(2^{1+2j}, 2^{4+2j})$. Then, as in section 2.2, we can join $\Gamma(s_j)$ to $\Gamma(s_{j+1})$ through a curve Γ'_j of length less than or equal to $c2^{2j}$, (c is a constant independent of j), entirely contained in a connected component $\mathcal{A}_c(2^{1+2j}/3, 32^{4+2j})$ of $\mathcal{A}(2^{1+2j}/3, 3\, 2^{4+2j})$. This curve Γ'_j must have ($N^{-2}g$)) length greater than or equal to the ($N^{-2}g$) length of Γ_j because Γ_j, (being a ray), minimizes the ($N^{-2}g$) length between any two of its points. Thus, we can write

$$\int_{s_j}^{s_{j+1}} \frac{1}{N(\Gamma(s))}\mathrm{d}s \leqslant \int_{s'_j}^{s'_{j+1}} \frac{1}{N\left(\Gamma'_j(s')\right)}\mathrm{d}s' \leqslant \frac{c2^{2j}}{\min\left\{N;\, \overline{\mathcal{A}}_c\left(2^{1+2j}/3,\, 3\, 2^{4+2j}\right)\right\}}. \tag{3.20}$$

Together with (3.19), we obtain

$$\mathrm{length}\left(\Gamma_j\right) \leqslant c\left[\frac{\max\left\{N;\, \Gamma_j\right\}}{\min\left\{N;\, \overline{\mathcal{A}}_c\left(2^{1+2j}/3,\, 32^{4+2j}\right)\right\}}\right]2^{2j}. \tag{3.21}$$

But from (2.16) we have

$$\frac{\max\left\{N;\, \Gamma_j\right\}}{\min\left\{N;\, \overline{\mathcal{A}}_c\left(2^{1+2j}/3,\, 32^{4+2j}\right)\right\}} \leqslant \frac{\max\left\{N;\, \overline{\mathcal{A}}_c\left(2^{1+2j}/3,\, 32^{4+2j}\right)\right\}}{\min\left\{N;\, \overline{\mathcal{A}}_c\left(2^{1+2j}/3,\, 32^{4+2j}\right)\right\}} \leqslant c', \tag{3.22}$$

where c' is independent of j. Thus, (3.17) follows. □

Proof of theorem 1.3. From the same definition of a static isolated system, we know that the spacetime outside a set (invariant under the Killing field) is

$$\mathbf{M} = \mathbb{R} \times \left(\mathbb{R}^3\backslash\mathbb{B}^3\right), \qquad \mathbf{g} = -N^2\mathrm{d}t^2 + g, \tag{3.23}$$

which is described by the data ($\mathbb{R}^3\backslash\mathbb{B}^3$; g, N). As the spacetime is geodesically complete at infinity, we can use proposition 3.3 to deduce that the metric N^2g is complete on $\mathbb{R}^3\backslash\mathbb{B}^3$. Theorem 1.3 in [8] then apples and asymptotic flatness follows.

(Remark: the concept of an isolated system used in [8] is the same as in this paper but with the extra assumption that N is bounded from below away from zero outside a compact set. As noted in [8], theorem 1.3 still holds if this hypothesis on N is replaced by the metric completeness of $N^2 g$.) □

Remark 3.4. If the matter model (which is always assumed to be compactly supported), satisfies the weak energy condition, then the conclusions of theorem 1.3 can be seen to follow only from the metric completeness of the static data. The geodesic completeness at infinity is unnecessary.

We can now prove theorem 1.4.

Proof of theorem 1.4. Suppose that a connected component of the *complement of a compact set* in Σ is diffeomorphic to $\mathbb{R}^3$ minus a closed ball. Then, as in the proof of theorem 1.3, this component has to be an AF end of Σ. If we prove that Σ has only one end, then the main theorem in [5] shows that Σ is diffeomorphic to $\mathbb{R}^3$ minus a finite set of open balls. The Israel [6]—Robinson [11]—Bunting–Masood-ul Alam [3] uniqueness theorem then applies and the solution is Schwarzschild. Let us prove then that Σ must have only one end.

We will proceed by contradiction. Assume then that Σ has more than one end. From now on, we work in a space $(\Sigma_\delta, N^{-2\epsilon}g)$ as in proposition 2.1 but with $\epsilon < \sqrt{2} - 1$.

The end that was AF (and had Schwarzschildian fall off) for g is also AF for $N^{-2\epsilon}g$. On this end consider large ('almost round') embedded spheres S. On these spheres we have $|\nabla N|_{N^{2-\epsilon}g} \lesssim 1/\text{area}(S)$, while for the mean curvature θ_S (with respect to the outward unit normal n) we have $\theta_S \approx 2\sqrt{4\pi/\text{area}(S)}$. Hence, one can clearly take an embedded sphere S sufficiently far away that

$$\theta_S - (1 + \epsilon)\frac{n(N)}{N} > 0 \tag{3.24}$$

at every point of S. We work with such S below. The particular combination (3.24) will be relevant. The sphere S divides Σ_δ into two connected components. Denote by Σ'_δ the closure of the connected component of $\Sigma_\delta \backslash S$ containing $\partial\Sigma$. We have $\partial\Sigma'_\delta = \partial\Sigma \cup S$ and, more importantly, Σ'_δ contains at least one more end. Since $\partial\Sigma_\delta$ is strictly convex, we can construct a geodesic ray $\gamma(s)$, $s \geqslant 0$, in $\Sigma'_\delta \backslash \partial\Sigma$ and with the following properties:

1. $\gamma(s)$ starts at S and perpendicular to it,
2. $\gamma(s)$ diverges through and ends in Σ'_δ as $s \to \infty$,
3. $\text{dist}_{N^{-2\epsilon}g}(\gamma(s), S) = s$ for all $s \geqslant 0$.

These properties imply that the expansion $\theta(s)$, along the geodesic $\gamma(s)$, of the congruence of geodesics emanating perpendicularly to S must remain finite for all s (i.e. $\theta(s) > -\infty$ for all $s \geqslant 0$). If not then there is a focal point on γ after which property 3 fails. We will prove now that indeed $\theta(s) = -\infty$ for some $s > 0$, thus reaching a contradiction.

Let

$$m(s) = \theta(s) + (1 + \epsilon)\frac{N'(s)}{N(s)}, \tag{3.25}$$

where $N(s) = N(\gamma(s))$ and $N'(s) = \mathrm{d}N(\gamma(s))/\mathrm{d}s$. At $s = 0$, m is equal to minus the left-hand side of (3.24), and is therefore negative (note that $\gamma'(0) = -n$). On the other hand, as we explained in section 2.1, if $\epsilon < \sqrt{2} - 1$, then the Bakry–Emery Ricci tensor

$$\mathrm{Ric}^c_f = \mathrm{Ric} + \nabla\nabla f - \frac{1}{c}\nabla f\,\nabla f \qquad (3.26)$$

is zero, where $f = (1+\epsilon)\ln N$ and $1/c = (1-2\epsilon-\epsilon^2)/(1+\epsilon)^2$. Now, it is shown in [12] (appendix A) that $m(s)$ satisfies the differential inequality

$$m' \leqslant -\frac{m^2}{2\;+\;c}. \qquad (3.27)$$

Thus, if $m(0) < 0$, then there is $s' > 0$ such that $m(s') = -\infty$. But as $N'(s)/N(s)$ is finite for all s, then we must have $\theta(s') = -\infty$. □

Remark 3.5. If the complement of a compact set in Σ is diffeomorphic to $\mathbb{R}^3\setminus\overline{\mathbb{B}^3}$ and $(\Sigma; g)$ is metrically complete, then the solution is also Schwarzschild (i.e. the geodesic completeness of the spacetime at infinity is unnecessary). To see this, observe first that N cannot go uniformly to zero on the end of Σ because this would violate the maximum principle (N is harmonic and is zero only on $\partial\Sigma$). By proposition 3.3 N is then bounded away from zero on the end and asymptotic flatness follows.

Remark 3.6. It is easy to show that propositions 3.1, 3.2 and 3.3 hold true when $\Sigma \approx \mathcal{S}\times\mathbb{R}^+$ with $\mathcal{S}$ being a compact two-surface of arbitrary genus (proposition 3.1 corresponds to $\mathcal{S} = \mathbb{S}^2$). This could be of interest in further studies.

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